

A conditionally exactly solvable generalization of the potential step

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Abstract

Motivated by the interest in non-relativistic quantum mechanics for determining exact solutions to the Schrödinger equation we give two potentials that are conditionally exactly solvable. The two potentials are partner potentials and we obtain that each linearly independent solution of the Schrödinger equation includes two hypergeometric functions. Furthermore we calculate their reflection and transmission amplitudes. Finally we discuss some additional properties of these potentials.

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1 Introduction

In physics a research line is the search and detailed study of systems for which we can solve exactly their equations of motion. We expect that the exactly solvable systems play a significant role since they are useful approximations to more complex systems and they allow us to make a more detailed analysis of the physical phenomena. Furthermore using these exactly solvable systems we can infer some details about the behavior of more complicated physical systems. In non-relativistic quantum mechanics the search of potentials for which we can solve exactly the Schrödinger equation is widely studied. See [1]–[13] for an incomplete list of references.

Several methods have been used to find exact solutions to the Schrödinger equation. We mention the factorization method [1], [2], [3] and the method based

on supersymmetric quantum mechanics (SUSYQM) [4]–[10]. Other widely used procedures are based on the Darboux transformation [11], [12] and on the point canonical transformations [13].

Recently we solve exactly the Schrödinger equations for the partner potentials [14]

$$\hat{V}_{\pm} = \frac{m^2}{x} \pm \frac{m}{2x^{3/2}}, \quad (1)$$

where m is a constant. In Ref. [14] we show that each of their linearly independent solutions includes a sum of two confluent hypergeometric functions. More recently, for \tilde{V}_0 a constant, in Ref. [15] it is showed that for the inverse square root potential

$$V_{1/2} = \frac{\tilde{V}_0}{\sqrt{x}}, \quad (2)$$

its exact solutions have a similar mathematical form to those found previously in Ref. [14], that is, each linearly independent solution is given by a linear combination of two confluent hypergeometric functions. Furthermore in Ref. [16] it is showed that for the sum of the potentials (1) and (2) it is possible to solve exactly the Schrödinger equation. In this case, in a similar way to the potentials (1) and (2), each linearly independent solution is a sum with non-constant prefactors of two confluent hypergeometric functions.

Based on the method of Ref. [14] in this work we present two potentials for which we solve exactly their Schrödinger equations when the parameters satisfy a condition, thus the potentials that we study are conditionally exactly solvable (CES) in the broad sense of the concept recently introduced in Ref. [16]. Furthermore for each potential the linearly independent solutions of the Schrödinger equation include two hypergeometric functions, hence our results extend those of Refs. [14], [15], [16]. The two potentials are partner potentials in the standard language of the SUSYQM. We see that the shapes of the potentials remind us to the step potential (see Figs. 1 and 3 below), thus we believe that the potentials can be useful to study scattering problems where the step-like potentials are useful models [17].

In what follows we expound in detail the method used to find the exact solutions, but here we notice that it works simultaneously with the two Schrödinger equations of the partner potentials [14]. As far as we can see these two potentials do not appear in the references that enumerate the solvable potentials that are previously known [6]–[10], [18]–[25].

We organize this paper as follows. In Sect. 2 we present the partner potentials that we study throughout this work and we explore some of their properties. Furthermore we describe the method used to find the exact solutions to the Schrödinger equations of these potentials. In Sect. 3 we give the solutions in explicit form and we verify that the functions found in the previous section are solutions of the Schrödinger equations. In Sect. 4 we use the solutions previously found to calculate the reflection and transmission amplitudes of the studied potentials. We also determine their quasinormal frequencies. Finally in Sect. 5 we discuss some facts on these partner potentials.

2 Solution method

The purpose of this work is to show that the Schrödinger equation¹

$$\frac{d^2 Z}{dx^2} + \omega^2 Z = VZ, \quad (3)$$

is solvable for the partner potentials

$$V_{\pm}(x, m) = m^2 \frac{e^x}{e^x + 1} \mp \frac{m}{2} \frac{e^{x/2}}{(e^x + 1)^{3/2}} = W^2 \pm \frac{dW}{dx}, \quad (4)$$

where m is a constant, the variable $x \in (-\infty, +\infty)$, and the superpotential W is equal to²

$$W(x, m) = -m \sqrt{\frac{e^x}{e^x + 1}} = -\frac{m}{(1 + e^{-x})^{1/2}}. \quad (5)$$

The potentials (4) behave as

$$\lim_{x \rightarrow +\infty} V_{\pm} = m^2, \quad (6)$$

and for $m > 0$

$$\lim_{x \rightarrow -\infty} V_+ = 0^-, \quad \lim_{x \rightarrow -\infty} V_- = 0^+, \quad (7)$$

where 0^- (0^+) means that the potential goes to zero taking negative (positive) values. For $m < 0$, the potentials V_+ and V_- change their places in the formulas (7). It is convenient to notice that as $x \rightarrow +\infty$ the potentials V_{\pm} go to m^2 and the subleading terms are of the form $\exp(-x)$, whereas they decay exponentially to zero as $x \rightarrow -\infty$. We plot the potentials (4) in Fig. 1. Furthermore from the shape of the potentials V_{\pm} we expect that their spectra are continuous [17], [26].

For the superpotential (5) we obtain

$$W_+ = \lim_{x \rightarrow +\infty} W = -m, \quad W_- = \lim_{x \rightarrow -\infty} W = 0. \quad (8)$$

As $x \rightarrow -\infty$, for $m > 0$ the superpotential W goes to zero taking negative values, whereas for $m < 0$ it goes to zero taking positive values. Since the exponential function satisfies $e^x > 0$ for $x \in (-\infty, +\infty)$ we point out that the superpotential (5) does not cross the x axis, hence it is positive for $m < 0$ and negative for $m > 0$. We plot the superpotential (5) in Fig. 2. In what follows we assume that $m > 0$.

¹In contrast to the common usage and for simplifying some of the following mathematical expressions, we write the energy E as ω^2 , that is, $E = \omega^2$.

²The Schrödinger equations for the partner potentials derived from the superpotential $\hat{W} = -me^{x/2}(\mathcal{A}e^x + \mathcal{B})^{-1/2}$ with $\mathcal{A} > 0$, $\mathcal{B} > 0$ reduce to those of the potentials for the superpotential (5) if we make the change of variable $y = \ln(\mathcal{A}/\mathcal{B}) + x$ and we change the parameter m by $m/\mathcal{A}^{1/2}$.

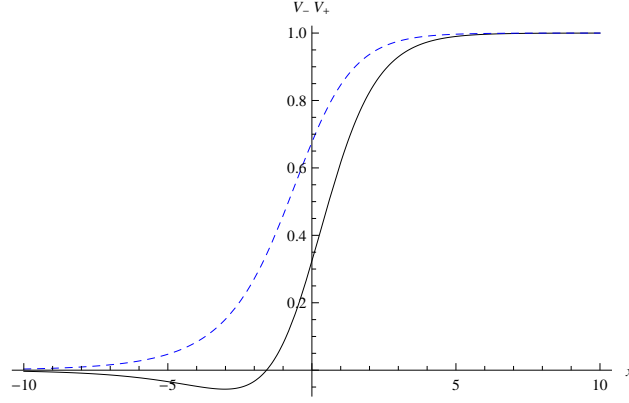


Figure 1: Plots of the potentials V_+ (solid line) and V_- (broken line) for $m = 1$.

Taking into account that $e^x = e^{x/2}/e^{-x/2}$ we obtain that the superpotential (5) becomes

$$W = -\mu(1 + \tanh(x/2))^{1/2}, \quad (9)$$

with $\mu = m/\sqrt{2}$, and the potentials (4) transform into

$$V_{\pm} = \mu^2(1 + \tanh(x/2)) \mp \frac{\mu(1 - \tanh(x/2))^{1/2}}{4 \cosh(x/2)}. \quad (10)$$

Comparing with the potential step [17]

$$V_S(x) = \frac{V_0}{2} \left(1 + \tanh\left(\frac{x}{2\alpha}\right) \right), \quad (11)$$

where V_0 and α are constants, we see that the potentials (10), and therefore the potentials (4), are a generalization of the potential step [17]. It is convenient to notice that the potential step can not be obtained as a limit of the partner potentials V_{\pm} .

Among the solvable potentials there are several examples for which it is possible to find the exact solutions only when the parameters satisfy some constraints. These potentials are called conditionally exactly solvable (CES) potentials [9], [16], [23]–[25]. The algebraic form of some known CES potentials recall us to the potentials (4), for example, the CES potential [9], [25]

$$V_I(x) = \frac{a_I}{1 + e^{-2x}} - \frac{b_I}{(1 + e^{-2x})^{1/2}} - \frac{3}{4(1 + e^{-2x})^2}, \quad (12)$$

whose superpotential takes the form

$$W_I(x) = \frac{p_I}{(1 + e^{-2x})^{1/2}} - \frac{1}{2(1 + e^{-2x})} - \sqrt{\epsilon_0}, \quad (13)$$

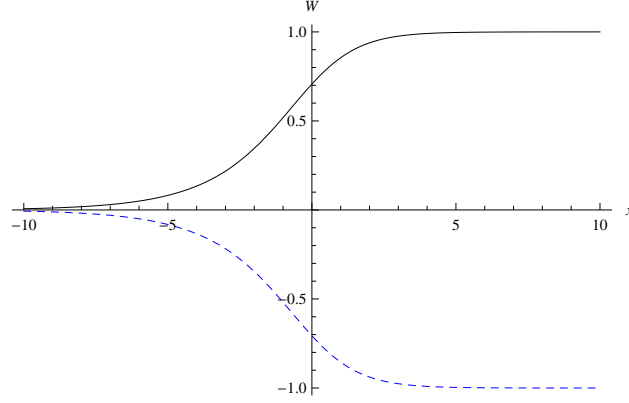


Figure 2: Plots of the superpotential W for $m = 1$ (broken line) and for $m = -1$ (solid line).

where the constants a_I , b_I , p_I , and ϵ_0 satisfy some constraints [9], [25]. Another example is the CES potential [9], [25]

$$V_{II}(x) = \frac{a_{II}}{1 + e^{-2x}} - \frac{b_{II}e^{-x}}{(1 + e^{-2x})^{1/2}} - \frac{3}{4(1 + e^{-2x})^2}, \quad (14)$$

with superpotential

$$W_{II}(x) = p_{II} + \frac{1}{2(1 + e^{2x})} - \frac{q_{II}}{(1 + e^{2x})^{1/2}}. \quad (15)$$

In a similar way to the previous CES potential, the constants a_{II} , b_{II} , p_{II} , and q_{II} satisfy some constraints [9], [25].

Owing to the constants multiplying the factors $e^x/(e^x+1)$ (m^2) and $e^{x/2}/(e^x+1)^{3/2}$ ($\pm m/2$) satisfy

$$\frac{1}{4}m^2 - \left(\pm \frac{m}{2}\right)^2 = 0, \quad (16)$$

we find that the potentials (4) are CES in the sense of Ref. [16], that is, we must understand that the parameters multiplying the factors $e^x/(e^x+1)$ and $e^{x/2}/(e^x+1)^{3/2}$ of V_{\pm} can not be varied independently. Also we point out that the superpotential (5) is not analyzed in Refs. [9], [25], since we can not choose the values of the parameters a_I , b_I , p_I , and ϵ_0 (a_{II} , b_{II} , p_{II} , and q_{II}) to simplify the superpotential W_I (W_{II}) to our superpotential (5).

As in Ref. [14], to solve the Schrödinger equations of the potentials (4) we write

$$\begin{aligned} \frac{d^2 Z_+}{dx^2} + \omega^2 Z_+ &= \left(W^2 + \frac{dW}{dx} \right) Z_+, \\ \frac{d^2 Z_-}{dx^2} + \omega^2 Z_- &= \left(W^2 - \frac{dW}{dx} \right) Z_-, \end{aligned} \quad (17)$$

and note that for $\omega \neq 0$ we can transform these equations into

$$\begin{aligned} \left(\frac{d}{dx} + W \right) \frac{1}{i\omega} \left(\frac{d}{dx} - W \right) Z_+ &= i\omega Z_+, \\ \left(\frac{d}{dx} - W \right) \frac{1}{i\omega} \left(\frac{d}{dx} + W \right) Z_- &= i\omega Z_-, \end{aligned} \quad (18)$$

from which we obtain that the functions Z_+ and Z_- satisfy the coupled equations

$$\left(\frac{d}{dx} - W \right) Z_+ = i\omega Z_-, \quad \left(\frac{d}{dx} + W \right) Z_- = i\omega Z_+. \quad (19)$$

Defining the functions R_1 and R_2 by $Z_+ = R_1 + R_2$, $Z_- = R_1 - R_2$, we find that Eqs. (19) become

$$\frac{d\tilde{R}_1}{dx} - i\omega\tilde{R}_1 = iW\tilde{R}_2, \quad \frac{d\tilde{R}_2}{dx} + i\omega\tilde{R}_2 = -iW\tilde{R}_1, \quad (20)$$

where we take $\tilde{R}_1 = e^{i\pi/4}R_1$, $\tilde{R}_2 = e^{-i\pi/4}R_2$. Defining the variable z by³

$$z = \frac{e^x}{e^x + 1}, \quad (21)$$

we get that for the superpotential (5) the coupled differential equations (20) take the form

$$\begin{aligned} z(1-z) \frac{d\tilde{R}_1}{dz} - i\omega\tilde{R}_1 &= -imz^{1/2}\tilde{R}_2, \\ z(1-z) \frac{d\tilde{R}_2}{dz} + i\omega\tilde{R}_2 &= imz^{1/2}\tilde{R}_1. \end{aligned} \quad (22)$$

From this coupled system of ordinary differential equations in a straightforward way we obtain that the functions \tilde{R}_1 and \tilde{R}_2 are solutions of the decoupled differential equations

$$\begin{aligned} \frac{d^2\tilde{R}_j}{dz^2} + \left(\frac{1/2}{z} - \frac{1}{1-z} \right) \frac{d\tilde{R}_j}{dz} + \frac{(i\omega/2)\epsilon}{z^2(1-z)} \tilde{R}_j \\ + \frac{\omega^2}{z^2(1-z)^2} \tilde{R}_j - \frac{m^2}{z(1-z)^2} \tilde{R}_j = 0, \end{aligned} \quad (23)$$

with $j = 1, 2$, $\epsilon = 1$ for \tilde{R}_1 , and $\epsilon = -1$ for \tilde{R}_2 . Taking the functions \tilde{R}_j as

$$\tilde{R}_j = z^{C_j} (1-z)^{B_j} \bar{R}_j, \quad (24)$$

with the quantities C_j and B_j being solutions of the algebraic equations⁴

$$C_j^2 - \frac{C_j}{2} + \frac{i\omega\epsilon}{2} + \omega^2 = 0, \quad B_j^2 + \omega^2 - m^2 = 0, \quad (25)$$

³Notice that z varies over the range $0 < z < 1$.

⁴We note that the parameter ϵ only appears in Eqs. (25) that determine the constants C_1 and C_2 . In the equations for the constants B_1 and B_2 this parameter does not appear.

we get that the functions \bar{R}_j satisfy

$$z(1-z)\frac{d^2\bar{R}_j}{dz^2} + \left(2C_j + \frac{1}{2} - \left(2C_j + 2B_j + \frac{3}{2}\right)z\right)\frac{d\bar{R}_j}{dz} - \left(m^2 + 2B_jC_j + C_j + \frac{B_j}{2} - 2\omega^2 - \frac{i\omega\epsilon}{2}\right)\bar{R}_j = 0, \quad (26)$$

which are hypergeometric type differential equations [27]–[30]

$$z(1-z)\frac{d^2f}{dz^2} + (c - (a+b+1)z)\frac{df}{dz} - abf = 0, \quad (27)$$

with parameters

$$a_j = C_j + B_j + \frac{1}{2}, \quad b_j = C_j + B_j, \quad c_j = 2C_j + \frac{1}{2}. \quad (28)$$

Assuming that the parameters c_j are not integers to discard the solutions including logarithmic terms [27]–[30] we obtain that the functions \bar{R}_j are equal to

$$\begin{aligned} \bar{R}_j &= G_j {}_2F_1(a_j, b_j; c_j; z) \\ &+ H_j z^{1-c_j} {}_2F_1(a_j - c_j + 1, b_j - c_j + 1; 2 - c_j; z), \end{aligned} \quad (29)$$

where ${}_2F_1(a, b; c; z)$ denotes the hypergeometric function [27]–[30] and G_j, H_j are constants. Therefore the functions \tilde{R}_1 and \tilde{R}_2 take the form

$$\begin{aligned} \tilde{R}_j &= z^{C_j}(1-z)^{B_j} [G_j {}_2F_1(a_j, b_j; c_j; z) \\ &+ H_j z^{1-c_j} {}_2F_1(a_j - c_j + 1, b_j - c_j + 1; 2 - c_j; z)]. \end{aligned} \quad (30)$$

Considering the definitions of \tilde{R}_j and Z_{\pm} , from the previous formulas we obtain the solutions of the Schrödinger equations (3) with the potentials (4) (see Eqs. (39) and (40) below).

3 Checking the solutions

Here we verify that the functions (30) give the exact solutions to the Schrödinger equations of the potentials (4). From the expressions (30) we obtain that these functions satisfy

$$\frac{d}{dz} \left(z(1-z)\frac{d\tilde{R}_j}{dz} \right) = \left(\frac{B_j^2}{1-z} + \frac{C_j^2 - C_j/2}{z} \right) \tilde{R}_j + \frac{1}{2}(1-z)\frac{d\tilde{R}_j}{dz}. \quad (31)$$

Considering that $Z_{\pm} = e^{-i\pi/4}(\tilde{R}_1 \pm i\tilde{R}_2)$, from the expressions (25) and (31) we get that the functions Z_{\pm} fulfill

$$\frac{d}{dz} \left(z(1-z)\frac{dZ_{\pm}}{dz} \right) + \left(\frac{\omega^2 - m^2}{1-z} + \frac{\omega^2}{z} \pm \frac{m}{2} \frac{1}{z^{1/2}} \right) Z_{\pm} = 0. \quad (32)$$

We can verify that Eqs. (32) are the Schrödinger equations for the potentials (4) but in the variable z . Therefore the functions Z_{\pm} are the exact solutions to the Schrödinger equations of the potentials (4).

It is convenient to notice that Eqs. (22) impose some restrictions on the values of the constants G_j and H_j . To discuss this fact in what follows we take B_j and C_j in the form

$$B_1 = B_2 = \sqrt{m^2 - \omega^2}, \quad C_2 = C_1 + \frac{1}{2} = i\omega + \frac{1}{2}, \quad (33)$$

and therefore the parameters a_j , b_j , and c_j are equal to

$$\begin{aligned} a_2 &= \sqrt{m^2 - \omega^2} + i\omega + 1 = b_1 + 1, \\ b_2 &= \sqrt{m^2 - \omega^2} + i\omega + \frac{1}{2} = a_1, \\ c_2 &= 2i\omega + \frac{3}{2} = c_1 + 1. \end{aligned} \quad (34)$$

Taking the solution \tilde{R}_1 as

$$\tilde{R}_1 = G_1 z^{C_1} (1-z)^{B_1} {}_2F_1(a_1, b_1; c_1; z), \quad (35)$$

substituting this expression in Eqs. (22), we find that the function \tilde{R}_2 must be of the form

$$\tilde{R}_2 = \frac{i}{m} \frac{(a_1 - c_1)b_1}{c_1} G_1 z^{C_2} (1-z)^{B_2} {}_2F_1(a_2, b_2; c_2; z), \quad (36)$$

that is, the constants G_1 and G_2 satisfy $G_2 = i(a_1 - c_1)b_1 G_1 / (mc_1)$.

Similar results are valid for the solution

$$\tilde{R}_1 = H_1 (1-z)^{B_1} z^{C_1+1-c_1} {}_2F_1(a_1 - c_1 + 1, b_1 - c_1 + 1; 2 - c_1; z), \quad (37)$$

since Eqs. (22) produce that the function \tilde{R}_2 must take the form

$$\tilde{R}_2 = \frac{i(1-c_1)}{m} H_1 z^{C_2} (1-z)^{B_2} z^{1-c_2} {}_2F_1(a_2 - c_2 + 1, b_2 - c_2 + 1; 2 - c_2; z), \quad (38)$$

and we obtain that the constants H_1 and H_2 satisfy $H_2 = i(1-c_1)H_1/m$. Thus Eqs. (22) impose constraints on the values of the constants G_j and H_j .

From the previous results we get that for the Schrödinger equations of the potentials (4) their linearly independent solutions take the form

$$\begin{aligned} Z_{\pm}^I &= G_1 e^{-i\pi/4} [z^{C_1} (1-z)^{B_1} {}_2F_1(a_1, b_1; c_1; z) \\ &\mp \frac{(a_1 - c_1)b_1}{mc_1} z^{C_2} (1-z)^{B_2} {}_2F_1(a_2, b_2; c_2; z)] \end{aligned} \quad (39)$$

and

$$\begin{aligned} Z_{\pm}^{II} &= H_1 e^{-i\pi/4} [z^{C_1+1-c_1} (1-z)^{B_1} {}_2F_1(a_1 - c_1 + 1, b_1 - c_1 + 1; 2 - c_1; z) \\ &\mp \frac{1-c_1}{m} z^{C_2+1-c_2} (1-z)^{B_2} {}_2F_1(a_2 - c_2 + 1, b_2 - c_2 + 1; 2 - c_2; z)] \end{aligned} \quad (40)$$

It is convenient to notice that using the recurrence relations for the contiguous hypergeometric functions [28] and Eqs. (20) we can show in a direct way that the functions Z_{\pm}^I and Z_{\pm}^{II} are solutions of the Schrödinger equations for the potentials (4).

Furthermore, taking into account that the Wronskian of the linearly independent solutions to the hypergeometric equation (27) is [30]

$$\mathcal{W}_z[{}_2F_1(a, b; c; z), z^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; z)] = (1-c)z^{-c}(1-z)^{c-a-b-1}, \quad (41)$$

we find that the Wronskians of the solutions (39) and (40) are equal to (for $G_1 = H_1 = 1$)

$$\mathcal{W}_z[Z_{\pm}^I, Z_{\pm}^{II}] = \pm \frac{2\omega}{m}(1 - c_1). \quad (42)$$

As expected, they are constants.

4 Scattering

As we commented previously, the potentials (4) remind us to the step potential (see Fig. 1) and therefore we believe that the potentials V_{\pm} are appropriate to discuss scattering problems. Thus in what follows we calculate the reflection and transmission amplitudes of the partner potentials V_{\pm} . We first discuss the potential V_+ and then the potential V_- .

To begin, we notice that in the limits $x \rightarrow \pm\infty$ the variable z defined in the formula (21) behaves as

$$\begin{aligned} \lim_{x \rightarrow +\infty} z &\approx 1, & \lim_{x \rightarrow +\infty} (1-z) &\approx e^{-x}, \\ \lim_{x \rightarrow -\infty} z &\approx e^x, & \lim_{x \rightarrow -\infty} (1-z) &\approx 1. \end{aligned} \quad (43)$$

From the expressions (39) and (40) for the solutions Z_+^I and Z_+^{II} of the potential V_+ in the limit $x \rightarrow -\infty$ we obtain that they behave as

$$\lim_{x \rightarrow -\infty} Z_+^I \approx G_1 e^{-i\pi/4} e^{i\omega x}, \quad \lim_{x \rightarrow -\infty} Z_+^{II} \approx H_1 e^{-i\pi/4} \left(\frac{c_1 - 1}{m} \right) e^{-i\omega x}. \quad (44)$$

Taking into account Kummer's formula for the hypergeometric function [27]–[30],

$$\begin{aligned} {}_2F_1(a, b; c; x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b+1-c; 1-x) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-x)^{c-a-b} {}_2F_1(c-a, c-b; c+1-a-b; 1-x). \end{aligned} \quad (45)$$

we get that as $x \rightarrow +\infty$ the functions Z_+^I and Z_+^{II} behave in the form

$$\begin{aligned}\lim_{x \rightarrow +\infty} Z_+^I &\approx G_1 e^{-i\pi/4} \left[e^{-i\sqrt{\omega^2 - m^2}x} \frac{\Gamma(c_1)\Gamma(c_1 - a_1 - b_1)}{\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)} \left(1 + \frac{b_1}{m}\right) \right. \\ &\quad \left. + e^{i\sqrt{\omega^2 - m^2}x} \frac{\Gamma(c_1)\Gamma(a_1 + b_1 - c_1)}{\Gamma(a_1)\Gamma(b_1)} \left(1 + \frac{c_1 - a_1}{m}\right) \right], \\ \lim_{x \rightarrow +\infty} Z_+^{II} &\approx H_1 e^{-i\pi/4} \left[e^{-i\sqrt{\omega^2 - m^2}x} \frac{\Gamma(2 - c_1)\Gamma(c_1 - a_1 - b_1)}{\Gamma(1 - a_1)\Gamma(1 - b_1)} \left(1 + \frac{b_1}{m}\right) \right. \\ &\quad \left. + e^{i\sqrt{\omega^2 - m^2}x} \frac{\Gamma(2 - c_1)\Gamma(a_1 + b_1 - c_1)}{\Gamma(a_1 - c_1 + 1)\Gamma(b_1 - c_1 + 1)} \left(1 + \frac{c_1 - a_1}{m}\right) \right].\end{aligned}\quad (46)$$

In the previous formulas and in what follows we assume that $\omega > m$.

To solve the scattering problem for the potentials V_\pm we impose that the solutions of the Schrödinger equation (3) must behave in the form

$$Z_\pm \approx e^{i\omega x} + R^\pm e^{-i\omega x} \quad (47)$$

as $x \rightarrow -\infty$ and in the form

$$Z_\pm \approx T^\pm e^{i\sqrt{\omega^2 - m^2}x} \quad (48)$$

as $x \rightarrow +\infty$. In these formulas and in what follows R^+ and T^+ (R^- and T^-) denote the reflection and transmission amplitudes of the potential V_+ (V_-).

As is well known [26], if the solution of the Schrödinger equation takes the form

$$Z = E_I F^I + E_{II} F^{II}, \quad (49)$$

where E_I and E_{II} are constants, and the functions F^κ behave as

$$\begin{aligned}\lim_{x \rightarrow +\infty} F^\kappa &\approx a^{\kappa+} e^{i\omega x} + b^{\kappa+} e^{-i\omega x}, \\ \lim_{x \rightarrow -\infty} F^\kappa &\approx a^{\kappa-} e^{i\omega x} + b^{\kappa-} e^{-i\omega x},\end{aligned}\quad (50)$$

where $a^{\kappa+}$, $b^{\kappa+}$, $a^{\kappa-}$, $b^{\kappa-}$ are constants, and $\kappa = I, II$, then the reflection and transmission amplitudes are [18], [26]

$$R = \frac{b^{II+}b^{I-} - b^{I+}b^{II-}}{a^{I-}b^{II+} - b^{I+}a^{II-}}, \quad T = \frac{b^{II+}a^{I+} - b^{I+}a^{II+}}{a^{I-}b^{II+} - b^{I+}a^{II-}}. \quad (51)$$

For the solutions Z_+^I and Z_+^{II} , from the formulas (44) and (46) we deduce

that the quantities $a^{\kappa+}$, $b^{\kappa+}$, $a^{\kappa-}$, $b^{\kappa-}$ are

$$\begin{aligned}
a^{I-} &= 1, & a^{II-} &= 0, \\
b^{I-} &= 0, & b^{II-} &= \frac{c_1 - 1}{m}, \\
a^{I+} &= \frac{\Gamma(c_1)\Gamma(a_1 + b_1 - c_1)}{\Gamma(a_1)\Gamma(b_1)} \left(1 + \frac{c_1 - a_1}{m}\right), \\
a^{II+} &= \frac{\Gamma(2 - c_1)\Gamma(a_1 + b_1 - c_1)}{\Gamma(a_1 - c_1 + 1)\Gamma(b_1 - c_1 + 1)} \left(1 + \frac{c_1 - a_1}{m}\right), \\
b^{I+} &= \frac{\Gamma(c_1)\Gamma(c_1 - a_1 - b_1)}{\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)} \left(1 + \frac{b_1}{m}\right), \\
b^{II+} &= \frac{\Gamma(2 - c_1)\Gamma(c_1 - a_1 - b_1)}{\Gamma(1 - a_1)\Gamma(1 - b_1)} \left(1 + \frac{b_1}{m}\right).
\end{aligned} \tag{52}$$

Taking into account the previous expressions for the potential V_+ we obtain that its reflection and transmission amplitudes are equal to

$$\begin{aligned}
R^+ &= -\frac{b^{II-}b^{I+}}{b^{II+}} = \frac{1}{m} \frac{\Gamma(1 - a_1)\Gamma(1 - b_1)\Gamma(c_1)}{\Gamma(1 - c_1)\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)}, \\
T^+ &= \frac{b^{II+}a^{I+} - b^{I+}a^{II+}}{b^{II+}} = \left(1 + \frac{c_1 - a_1}{m}\right) \frac{\Gamma(1 - a_1)\Gamma(1 - b_1)}{\Gamma(1 - c_1)\Gamma(1 + c_1 - a_1 - b_1)}.
\end{aligned} \tag{53}$$

From these expressions we find that⁵

$$\begin{aligned}
R^+ R^{+*} &= \frac{\sinh(2\pi\omega - 2\pi\sqrt{\omega^2 - m^2})}{\sinh(2\pi\omega + 2\pi\sqrt{\omega^2 - m^2})}, \\
T^+ T^{+*} &= \frac{\omega}{\sqrt{\omega^2 - m^2}} \frac{2 \cosh(2\pi\omega) \sinh(2\pi\sqrt{\omega^2 - m^2})}{\sinh(2\pi\omega + 2\pi\sqrt{\omega^2 - m^2})}.
\end{aligned} \tag{54}$$

Furthermore, it is well known that for the Schrödinger equation (3) the quantity

$$Z^* \frac{dZ}{dx} - Z \frac{dZ^*}{dx}, \tag{55}$$

is a constant [17], [26]. For the approximate solutions (47) and (48) we get that the previous condition implies that the reflection and transmission amplitudes satisfy

$$R^+ R^{+*} + \frac{\sqrt{\omega^2 - m^2}}{\omega} T^+ T^{+*} = 1. \tag{56}$$

An straightforward calculation shows that the expressions (53) for the reflection and transmission amplitudes of the potential V_+ fulfill the condition (56).

⁵The asterisk denotes the complex conjugation.

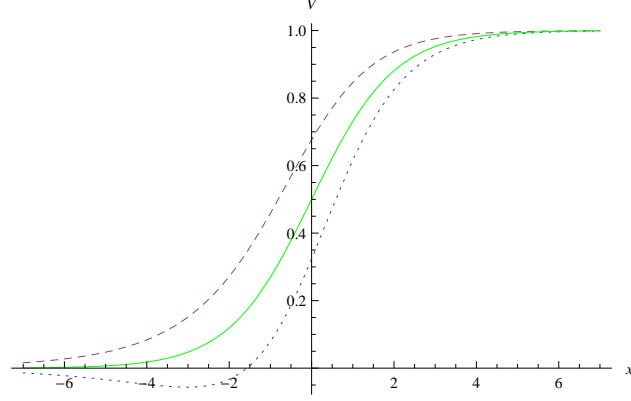


Figure 3: Plots of the potentials V_+ (dotted line), V_- (dashed line) and V_S (solid line) for $V_0 = m^2$, $\alpha = 1$, and $m = 1$.

A similar computation shows that for the potential V_- its reflection and transmission amplitudes take the form

$$\begin{aligned} R^- &= -\frac{1}{m} \frac{\Gamma(1-a_1)\Gamma(1-b_1)\Gamma(c_1)}{\Gamma(1-c_1)\Gamma(c_1-a_1)\Gamma(c_1-b_1)} = -R^+, \\ T^- &= \left(1 - \frac{c_1-a_1}{m}\right) \frac{\Gamma(1-a_1)\Gamma(1-b_1)}{\Gamma(1-c_1)\Gamma(1+c_1-a_1-b_1)} = \frac{\left(1 - \frac{c_1-a_1}{m}\right)}{\left(1 + \frac{c_1-a_1}{m}\right)} T^+. \end{aligned} \quad (57)$$

Simplifying the last expression we also find the relationship

$$T^+ = \frac{i\omega}{m + i\sqrt{\omega^2 - m^2}} T^-. \quad (58)$$

From these expressions for the amplitudes R^- and T^- we obtain that $R^- R^{-*} = R^+ R^{+*}$ and $T^- T^{-*} = T^+ T^{+*}$, that is, the amplitudes R^- and T^- of the potential V_- satisfy the condition (56).

As we commented before, the potentials V_{\pm} are partner potentials. Thus SUSYQM predicts a relationship between their reflection (transmission) amplitudes (see for example the formulas (16.7) and (16.8) of Ref. [8]). From these expressions for the potentials V_{\pm} we get

$$R^+ = -R^-, \quad T^+ = \frac{ik}{m + ik'} T^-, \quad (59)$$

where the quantities k and k' are equal to

$$k = \sqrt{\omega^2 - W_-^2} = \omega, \quad k' = \sqrt{\omega^2 - W_+^2} = \sqrt{\omega^2 - m^2}. \quad (60)$$

We see that the formulas (59) coincide with the previously obtained relationships (57) and (58). Notice that SUSYQM does not allow to calculate the full expression for the reflection and transmission amplitudes (53) and (57).

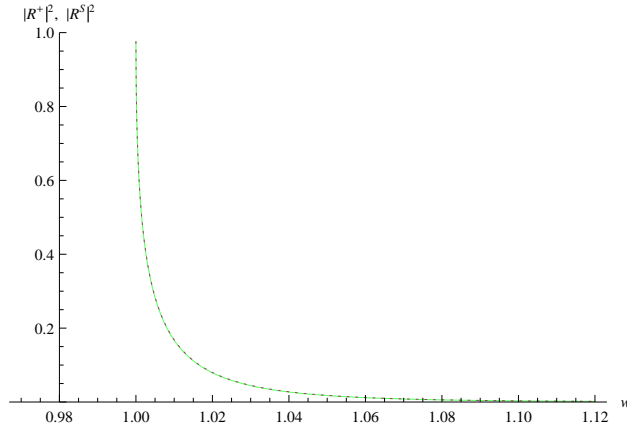


Figure 4: Plots of the reflection coefficients for $\omega > m$ of the potential V_+ (dotted line) and for the step potential V_S (solid line). We take $V_0 = m^2$, $\alpha = 1$, and $m = 1$.

We think that the partner potentials V_{\pm} generalize the potential step (11) and we observe that their shapes are similar (see Fig. 3). Since the reflection amplitude of the step potential (11) is equal to (in what follows for the potential step we take $V_0 = m^2$, $\alpha = 1$) [17]

$$R^S = \frac{\Gamma(2i\omega)\Gamma(-i\omega - i\sqrt{\omega^2 - m^2})\Gamma(1 - i\omega - i\sqrt{\omega^2 - m^2})}{\Gamma(-2i\omega)\Gamma(i\omega - i\sqrt{\omega^2 - m^2})\Gamma(1 + i\omega - i\sqrt{\omega^2 - m^2})}, \quad (61)$$

and therefore we get that its reflection coefficient takes the form

$$R^S R^{S*} = \frac{\sinh^2(\pi\omega - \pi\sqrt{\omega^2 - m^2})}{\sinh^2(\pi\omega + \pi\sqrt{\omega^2 - m^2})}. \quad (62)$$

In Fig. 4 we compare the reflection coefficient (54) of the potential V_+ and the reflection coefficient (62) of the potential step (11). We see that apparently their reflection coefficients are equal, but plotting their difference (see Fig. 5) we notice that they are slightly different, but this difference is not visible in Fig. 4.

As is well known, the potential step (11) does not have transmission resonances [31]. In a similar way from the expressions (53) and (57) we obtain that the potentials V_{\pm} do not have transmission resonances. Closely related to the transmission resonances are the quasinormal frequencies, defined as the complex frequencies corresponding to purely outgoing waves as $x \rightarrow \pm\infty$ [31]. Thus we find the quasinormal frequencies by looking for the complex numbers where the transmission amplitude becomes infinite [31]. The transmission amplitude for

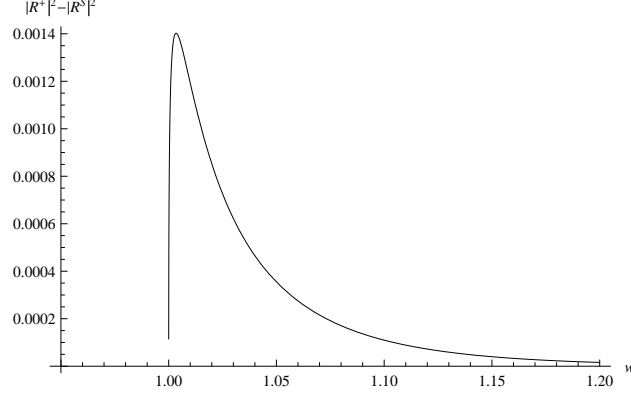


Figure 5: Difference of the reflection coefficients for $\omega > m$ of the potentials V_+ and V_S . In this plot we take $V_0 = m^2$, $\alpha = 1$, and $m = 1$.

the potential step (11) is equal to [17], [31]

$$T^S = \frac{\Gamma(1 - i\omega - i\sqrt{\omega^2 - m^2})\Gamma(-i\omega - i\sqrt{\omega^2 - m^2})}{\Gamma(-2i\omega)\Gamma(1 - 2i\sqrt{\omega^2 - m^2})}. \quad (63)$$

Therefore its quasinormal frequencies are determined by the poles of the numerator of the amplitude T^S located at [31]

$$1 - i\omega - i\sqrt{\omega^2 - m^2} = -n, \quad (64)$$

with $n = 0, 1, \dots$. Thus its quasinormal frequencies are equal to [31]

$$\omega_S = -\frac{i}{2} \left(n + 1 - \frac{m^2}{n + 1} \right). \quad (65)$$

In a similar way, from the expression (53) for the transmission amplitude T^+ of the potential V_+ we find that it transforms into

$$T^+ = \left(1 + \frac{c_1 - a_1}{m} \right) \frac{\sqrt{\pi}}{2^{-2i\omega - 2i\sqrt{\omega^2 - m^2}}} \frac{\Gamma(1 - 2i\omega - 2i\sqrt{\omega^2 - m^2})}{\Gamma(\frac{1}{2} - 2i\omega)\Gamma(1 - 2i\sqrt{\omega^2 - m^2})}. \quad (66)$$

We notice that it becomes infinite at the poles of the numerator, that is, the quasinormal frequencies of the potential V_+ are determined by

$$1 - 2i\omega - 2i\sqrt{\omega^2 - m^2} = -n, \quad n = 0, 1, 2, \dots \quad (67)$$

Thus the quasinormal frequencies of the potential V_+ are equal to

$$\omega_+ = -\frac{i}{4} \left(n + 1 - \frac{4m^2}{n + 1} \right). \quad (68)$$

A similar result is valid for the potential V_- . Hence we must add the potentials V_\pm to the list of Ref. [31] that enumerates the potentials with exactly calculated quasinormal frequencies.

Comparing the quasinormal frequencies (65) and (68) for the potential step and the potential V_+ , we find that their mathematical forms are similar, but there are relevant differences. For example, in the asymptotic limit $n \rightarrow \infty$ the quasinormal frequencies (65) behave as

$$\omega_S \approx -\frac{i}{2}n, \quad (69)$$

whereas the quasinormal frequencies (68) behave in the form

$$\omega_+ \approx -\frac{i}{4}n. \quad (70)$$

Furthermore in the quasinormal frequencies (65) and (68) the factor multiplying the parameter m is different.

5 Final remarks

To finish this work we note the following facts about the potentials (4).

1. From the expressions (4) for the potentials V_+ and V_- we notice that they fulfill $V_+(x, m) = V_-(x, -m)$. If $\hat{\alpha}_0, \hat{\alpha}_1$ are parameters, $R_0(\hat{\alpha}_0)$ is a function of $\hat{\alpha}_0$ and comparing the previous expressions with the formula $V_+(x, \hat{\alpha}_0) = V_-(x, \hat{\alpha}_1) + R_0(\hat{\alpha}_0)$, that defines the shape invariance condition for the partner potentials [32], we find that for the potentials (4), the quantities $\hat{\alpha}_0, \hat{\alpha}_1$, and $R_0(\hat{\alpha}_0)$ are equal to $\hat{\alpha}_0 = m, \hat{\alpha}_1 = -m = -\hat{\alpha}_0, R_0(\hat{\alpha}_0) = 0$, that is $\hat{\alpha}_1 = q\hat{\alpha}_0$ with $q = -1$. Therefore the potentials V_+ and V_- of the formulas (4) are multiplicative shape invariant [7]–[9]. Notice that we obtain the potentials (4) in closed form, in contrast, the previously known multiplicative shape invariant potentials that include a scaling of the parameters are given in series form [7].

The previous formula $\hat{\alpha}_1 = -\hat{\alpha}_0$ remind us a reflection, but on the parameters of the potential, which is different from the recent analyzed shape invariance with reflection transformations of Ref. [33], where they consider reflections of the coordinate and translations of the parameters.

2. We note that the indefinite integrals of the potentials (4) are equal to

$$\int V_\pm(x)dx = m^2 \ln(e^x + 1) \mp m \frac{e^{x/2}}{(e^x + 1)^{1/2}} + C_\pm, \quad (71)$$

where C_\pm are constants and the integrals

$$\int_{-\infty}^{+\infty} V_\pm(x)dx \quad (72)$$

diverge to $+\infty$.

3. Furthermore we notice that for the Schrödinger equations of the potentials (4) their linearly independent solutions include two hypergeometric functions with non constant coefficients (see the formulas (39) and (40)), in a similar way to the exact solutions studied previously in Refs. [14], [15], [16], but in the last references are involved confluent hypergeometric functions. We have not found a transformation that simplifies this sum of hypergeometric functions to a single hypergeometric function, but this problem must be analyzed carefully. Thus if the previous transformation is not possible, the potentials (4) are an example of those proposed but not given in explicit form in Ref. [34] whose linearly independent solutions include a sum of hypergeometric functions.

Undoubtedly we must study the use of the potentials V_{\pm} as a basis to generate new exactly solvable potentials [7].

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